Proof

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prove for 0≤p≤q≤0.5

$$q(1-q)[\frac{ln(\frac{q}{1-q}.\frac{1-p}{p})}{1-2q}+\frac{ln(\frac{1-q}{q})}{q-p}]\geq 1$$

Let, p = aq, where $0 \leq a \leq 1$ Then,

$$L.H.S = q(1-q)[\frac{1}{1-2q}.ln(\frac{1-aq}{a-aq})+\frac{1}{q(1-a)}ln(\frac{1-q}{q})]$$

Let,

$$Z = ln(\frac{1-aq}{a-aq})$$

$$Z = \frac{ln(1-aq)-ln(a-aq)}{(1-aq)-(a-aq)}.(1-a)$$

This is of the form

$$\frac{ln(y)-ln(x)}{y-x}$$

which is the slope of the line (x,ln(x)), (y,ln(y)). ln(x) is a concave function so,

$$x_1 \leq x_2 \implies f'(x_2) \leq slope \leq f'(x_1)$$

$$\implies Z \geq \frac{1-a}{1-aq}$$

$$\implies L.H.S \geq q(1-q)[\frac{1-a}{(1-aq).(1-2q)}+\frac{1}{q(1-a)}ln(\frac{1-q}{q})]$$

$$\implies L.H.S \geq (\frac{1-q}{1-aq}).\frac{q(1-a)}{1-2q}+\frac{1-q}{1-a}ln(\frac{1-q}{q})$$

Here,

$$1-aq \leq 1 \implies \frac{1}{1-aq} \geq 1$$

$$\implies L.H.S \geq \frac{q(1-q)(1-a)}{1-2q}+\frac{1-q}{1-a}ln(\frac{1-q}{q})$$

Let, b = 1-a $\implies$ $0 \leq b \leq 1$ and we get,

$$L.H.S \geq \frac{q(1-q)}{1-2q}.b + \frac{(1-q)ln(\frac{1-q}{q})}{b}$$

This is of the form,

$$Ax + \frac{B}{x}$$

and from lemma 0.1 we have it to be ≥ 1 when,

$$AB \geq \frac{1}{4} \; or \; (A + B \geq 1 \; and \; A \leq \frac{1}{2})$$

Let's, evaluate AB

$$AB = \frac{1}{1-2q}.q(1-q)^2 ln(\frac{1-q}{q})$$

This is a single variate (concave) function in $q \in [0, \frac{1}{2}]$ and

$$AB \geq \frac{1}{4}, \; for \; q \geq 0.127$$

$$\implies L.H.S \geq 1 \; for \; q \geq 0.127 \; or \; q \geq \frac{1}{4}$$

This proves the inequality for $q \geq \frac{1}{4}$
Now,

$$A(q) = \frac{q(1-q)}{1-2q}, \; A(\frac{1}{4}) = \frac{1}{4}$$

Since, A is an increasing function in $[0, \frac{1}{2}]$
$A \leq \frac{1}{2}$ for $q \leq \frac{1}{4}$ So, we can use the second condition from the lemma
Consider A+B

$$A + B = \frac{q(1-q)}{1-2q} + (1-q)ln(\frac{1-q}{q})$$

Now, this is a convex function in q for q $\in [0, \frac{1}{2}]$ The minimum value of this function is 1.1149, which is greater than 1

$$\implies A + B \geq 1, \; A \leq \frac{1}{2} \; for \; q \leq \frac{1}{4}$$

$$\implies L.H.S \geq 1 \; for \; q \leq \frac{1}{4}$$

Hence, True for $q \in [0, \frac{1}{2}]$

Lemma 0.1. *for $A \geq 0$ and $B \geq 0$*
if $AB \geq \frac{1}{4}$ or $(A + B \geq 1$ and $A \leq \frac{1}{2})$ then
$Ax + \frac{B}{x} \geq 1$ for $x \in [0, 1]$

Proof. Since, $x \geq 0$
The inequality $Ax + \frac{B}{x} \geq 1 \implies Ax^2 - x + B \geq 0$
Now Let, this quadratic $Ax^2 - x + B$ have two different roots α, β, $\alpha \leq \beta$
For the quadratic to be greater than 0 the roots can only be

1. Imaginary or a single root $\implies$ both roots above or touching the x-axis
2. $\alpha \geq 1 \implies$ both roots right of 1
3. $\beta \leq 0 \implies$ both roots left of 0

The roots are $\frac{1 \pm \sqrt{1-4AB}}{2A}$
$\alpha = \frac{1-\sqrt{1-4AB}}{2A}$, $\beta = \frac{1+\sqrt{1-4AB}}{2A}$

As we can see $\beta \geq 0$ hence, 3 cannot be possible

Now from 1. we get $AB \geq \frac{1}{4}$

from 2. we get,
$\frac{1-\sqrt{1-4AB}}{2A} \geq 1 \implies (1-2A) \geq \sqrt{1-4AB}$
$\implies (1-2A)^2 \geq 1 - 4AB$ and $A \leq \frac{1}{2}$ (other wise $-ve \geq +ve$)
$\implies 4A^2 + 4AB \geq A$ and $A \leq \frac{1}{2}$
$\implies A + B \geq 1$ and $A \leq \frac{1}{2}$ □